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COVERINGS BY MINIMAL TRANSVERSALS

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In this paper, we give counterexamples to the conjecture: “Every nonempty regular simple graph contains two disjoint maximal independent sets” [2, 7]. For this, we generalize this problem to the following: covering the set of vertices of a graph by minimal transversals. An equivalence of this last problem is given.

1. Introduction

The conjecture (C): “Every nonempty regular simple graph contains two disjoint maximal independent sets” has been raised by Berge [2] and independently by the author [7]. Cockayne and Hedetniemi have been also working on this problem [4, 5].

We give herein counterexamples to this conjecture by means of a generalization of this problem.

2. Definitions

In this paper, we will consider only finite, simple, loopless undirected graphs $G = (V(G), E(G))$ [$V(G)$ is the vertex set and $E(G)$ the edge set of G]. All main definitions can be found in [1].

For every graph G and every $S \subseteq V(G)$ we shall denote by G_S the subgraph of G induced by S .

$\gamma(G)$ designates the *chromatic number* of G .

$\delta(G)$ designates the *minimum degree* of a vertex of G .

$\Delta(G)$ designates the *maximum degree*.

$\forall A \subseteq V(G)$, A is said to be a *dominating set* of G iff

$$\forall v \in V(G) - A \exists a \in A \mid \{a, v\} \in E(G)$$

$\beta(G)$ designates the minimum cardinal of a dominating set of G .

$\forall A \subseteq V(G)$, A is said to be a *strongly dominating set* of G iff

$$\forall v \in V(G) \exists a \in A \mid \{a, v\} \in E(G).$$

It is easy to see that A is a strongly dominating set of G iff A is a dominating set of G such that G_A has no isolated vertex.

In a wider sense B is said to be a strongly dominating set of $X \subseteq V(G)$ iff $\forall v \in X \exists a \in A \mid (a, v) \in E(G)$. (X is said to be *strongly dominated* by B .)

3. Generalization

(a) A "strong sense" generalization would be the search for mutually disjoint maximal independent sets.

Let $b(G)$ be the maximum number of mutually disjoint maximal independent sets of a graph G . This parameter was defined by Cockayne and Hedetniemi [4] and by Grünbaum for the line graphs [6]. In [4] and [6] these authors give some results and conjectures. Some of these conjectures were invalidated in [8, 9].

(b) A "weak sense" generalization would be the search for coverings by minimal transversals. (Indeed conjecture C is equivalent to the following: "the set of vertices of a nonempty regular simple graph can be covered by two minimal transversals".)

Let $t(G)$ be the minimum number of minimal transversals required for covering $V(G)$. We shall give an equivalent formulation of this problem by regarding $t(G)$ as the chromatic number of a certain subgraph of G .

Theorem 3.1. *It is possible to cover the vertices of a graph G by t minimal transversals iff there exists a strongly dominating set A of G such that $\gamma(G_A) \leq t$. In other words*

$$t(G) = \min \gamma(G_A)$$

for A strongly dominating set of G .

Proof. (a) Let T_1, T_2, \dots, T_t be t minimal transversals such that:

$$T_1 \cup T_2 \cup \dots \cup T_t = V(G).$$

$\forall i \in \{1, 2, \dots, t\}$ let $S_i = V(G) - T_i$ a maximal independent set of G

$$S_1 \cap S_2 \cap \dots \cap S_t = \emptyset.$$

Therefore $A = S_1 \cup S_2 \cup \dots \cup S_t$ induces a subgraph G_A of G without isolated vertices. Moreover A is a dominating set of G (each S_i is a dominating set of G). Hence A is a strongly dominating set of G such that $\gamma(G_A) \leq t$.

(b) Let A be a strongly dominating set of G such that $\gamma(G_A) = k \leq t$. We can cover A by k maximal independent set of G_A : S'_1, S'_2, \dots, S'_k . $\forall i \in \{1, 2, \dots, k\}$ let $S'_i \supseteq S_i$ a maximal independent set of G . We claim $S'_1 \cap S'_2 \cap \dots \cap S'_k = \emptyset$. Otherwise $\forall v \in S'_1 \cap S'_2 \cap \dots \cap S'_k$, v would not be strongly dominated by A . $\forall i \in \{1, 2, \dots, k\}$ let $T_i = V(G) - S'_i$. T_i is a minimal transversal of G and $T_1 \cup T_2 \cup \dots \cup T_k = V(G)$. Hence, it is possible to cover $V(G)$ by $k \leq t$ minimal transversals.

4. Consequences

This result gives a tool for finding $t(G)$. For example, in Figs. 1 and 2 we check easily that $t(G_1) = t(G_2) = 3$ by looking for a strongly dominating set of the set of the vertices of degree 1 in G_1 and 2 in G_2 . Theorem 3.1 gives general results on $t(G)$. In particular we have (for proofs see [9]):

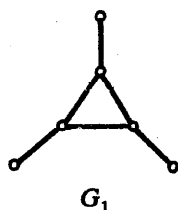


Fig. 1.

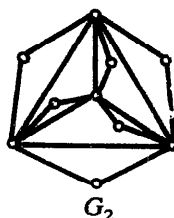


Fig. 2.

Corollary 4.1.

$$t(G) \leq \gamma(G),$$

$$t(G) \leq \max \{2, \Delta(G)\},$$

$$t(G) \leq \max \{2, \beta(G)\}$$

for every graph G without isolated vertices.

Let \mathcal{F} the family of: (a) line graphs, and more generally graphs without subgraph isomorphic to $K_{1,3}$; (b) complementary graphs of line graphs; (c) graphs of maximum degree $\Delta(G) \leq 3$.

Corollary 4.2.

$$t(G) \leq \max \left\{ 2, \frac{\Delta(G)}{\delta(G)} \right\}$$

for every graph $G \in \mathcal{F}$.

In particular every regular graph $G \in \mathcal{F}$ satisfies to the conjecture C.

For general graphs, we shall see that there is no relation between $t(G)$ and $\Delta(G)/\delta(G)$ and that it is possible to construct regular graphs G with any $t(G)$.

5. Counterexamples to the conjecture C

Theorem 5.1. $\forall p \in \mathbb{N}^+$ there exists a regular graph G_p such that $t(G_p) > p$.

Proof. Let G_p be the following graph:

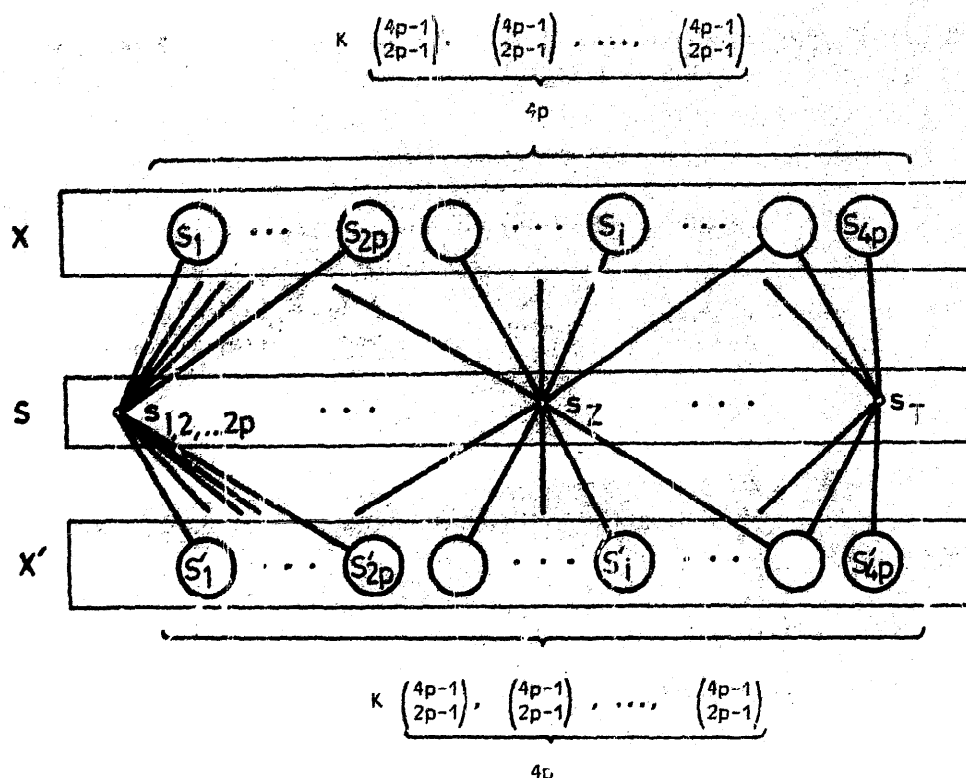


Fig. 3.

$$V(G_p) = X \cup X' \cup S,$$

$$X = s_1 \cup s_2 \cup \dots \cup s_{4p},$$

$$X' = s'_1 \cup s'_2 \cup \dots \cup s'_{4p}.$$

$\forall i \in \{1, 2, \dots, 4p\}$ s_i (resp. s'_i) is an independent set of $\binom{4p-1}{2p-1}$ vertices. $\forall x \in s_i$ (resp. s'_i), $\forall y \in s_j$ (resp. s'_j) $i \neq j \Leftrightarrow \{x, y\} \in E(G_p)$

$$S = \{s_Z : Z \subset \{1, 2, \dots, 4p\}, |Z| = 2p\}.$$

S is an independent set of G_p with $\binom{4p}{2p}$ vertices.

$$\forall v \in V(G_p) \quad \{s_Z, v\} \in E(G_p) \Leftrightarrow v \in s_i \text{ (resp. } s'_i) \text{ and } i \in Z.$$

(a) It is easy to check that G_p is a regular graph of degree d

$$d = 4p \binom{4p-1}{2p-1} = 2p \binom{4p}{2p}.$$

(b) Let A be a strongly dominating set of G_p . A contains a minimal strongly dominating set B of S . Suppose that $\gamma(G_{p_A}) \leq p$. Hence $\gamma(G_{p_B}) \leq p$. Therefore:

$$B \cap X \subseteq s_i \cup s_j \cup \dots \cup s_k \quad \text{with} \quad |\{i, j, \dots, k\}| \leq p.$$

$$B \cap X' \subseteq s'_u \cup s'_v \cup \dots \cup s'_t \quad \text{with} \quad |\{u, v, \dots, t\}| \leq p.$$

Let $Y = \{i, j, \dots, k\} \cup \{u, v, \dots, t\}$

$$|Y| \leq 2p.$$

Let $Z = \{1, 2, \dots, 4p\} - Y$

$$|Z| \geq 2p.$$

Let $U \subseteq Z$ $|U| = 2p$. $s_U \in S$ is not strongly dominated by B . Therefore B is not a strongly dominating set of S . This leads to a contradiction. The proof is achieved.

Hence, for $p = 2$ we obtain a regular graph of degree $d = 280$ and with 630 vertices which satisfy the conjecture C.

In a similar way, it is possible to obtain a counterexample of lower degree but with more vertices ($d = 252$, $|V| = 840$).

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